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Evaluation of single-fibre strength distribution from fibre bundle strength

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A new modified Weibull distribution function has been suggested for analysing the strength of fibres used as reinforcements for advanced composites. The function provides an upper and a lower strength limit and is characterized by two shape and two location parameters. A method for determining the parameters of this distribution from the analysis of the tensile curves of fibre bundles has also been developed. Application of the method to the experimental results on Thornel-300 carbon fibres shows that the shape parameters become modified in the case of bundles.

1. Introduction

Reinforcing fibres such as glass or carbon exhibit a very broad tensile strength distribution. This is usually attributed to the pre-existing flaws in these fibres, especially on fibre surfaces. Scatter in the sizes of these initial flaws accounts for the scatter in strength. Current theories of composite strength [1, 2] usually require an accurate estimation of the fibre strength and its distribution at short lengths of the order of the critical length for stress transfer (~ 0.5 mm or less). Since it is difficult to obtain reliable data from experimental measurements at these short lengths, the strength is usually obtained from an extrapolation of the mean strength and strength distribution data obtained at long lengths by using the Weibull distribution function. However, such an extrapolation tends to overestimate the strength value at short gauge lengths [3–6], and it has been concluded that a single Weibull distribution is inconsistent with the experimental data [7]. Similar observations have also been made by Olshansky *et al.* [8] and Kalish *et al.* [9] in analysing the strength of optical glass fibres.

Chi *et al.* [10] pointed out the experimental difficulties associated with the measurements of single-fibre strength to obtain reliable data. They derived a theoretical expression for the load–strain (P–ε) relationship for a bundle of fibres, assuming the strength of single fibres to have a unimodal Weibull distribution. By analysing the characteristics of the P–ε expression they developed methods for determining the parameters of the Weibull distribution for single fibre strength.

The present author has analysed the limitations of the unimodal Weibull distribution and proposed a new modified Weibull distribution function [11, 12] for the analysis of the strength of brittle fibres. In this paper, the same function has been used in deriving the P–ε expression for a bundle of fibres following the method given by Chi *et al.* [10]. The parameters of the proposed distribution for single-fibre strength are then determined following one of the methods given by Chi *et al.* [10]. The validity of the method is verified by analysing the strength of carbon fibres.

2. Analysis

2.1. Strength distribution function

Chi *et al.* [10] made the following assumptions in deriving their relations.

1. The single fibre strength distribution follows the Weibull distribution given by

$$F(\sigma) = 1 - \exp\left[-L(\sigma/\sigma_0)^m\right] \tag{1}$$

where $F(\sigma)$ is the failure probability of a single fibre of length L under an applied stress no greater than σ; σ_0 and m are the scaling parameter and Weibull modulus, respectively.

2. For a single fibre the applied stress σ and strain ε follow Hooke's law up to fracture:

$$\sigma = E_f \varepsilon \tag{2}$$

where E_f is the fibre elastic modulus.

3. The applied load is distributed uniformly among the surviving fibres at any instant during a bundle tensile test.

In the following derivation the last two assumptions remain the same; however, the flaw distribution function is modified based on the following argument. Statistically, the best-fitting distribution for any strength data can be obtained by calculating the standardized coefficients of kurtosis and skewness of the strength data [13]. From such an analysis of numerous sets of strength data for brittle solids, Snowden [13] has concluded that the beta distribution rather than the Weibull distribution describes the brittle-solid strength data best. As an example, an analysis of single carbon fibre strength data (Thornel-300) tested at a gauge length of 60 mm, reported by Chi and Chou [14], is shown in Fig. 1. The values of standardized coefficients of skewness square and kurtosis, as calculated from the various moments of this distribution, are obtained as 0.296 and 2.936, respectively, and the corresponding distribution is obtained as the beta distribution from Fig. 2 of Snowden [13]. Analysis of the data in terms of the Pearson system [15] of

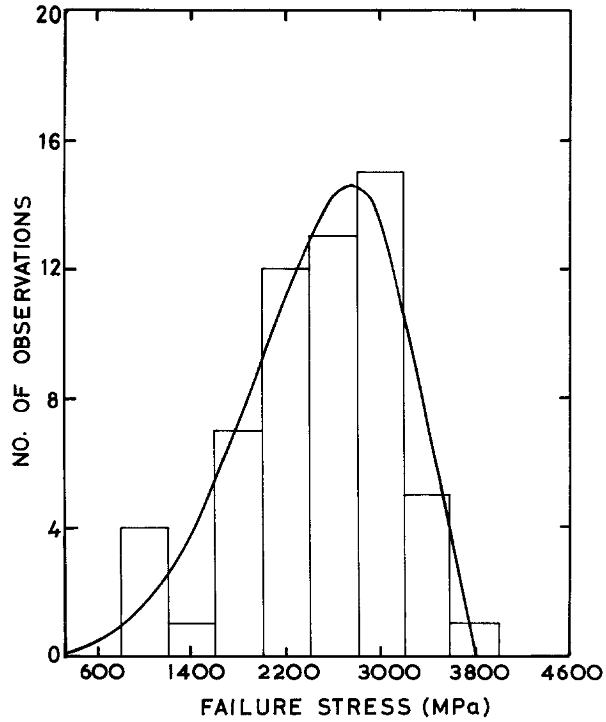

Figure 1 Tensile strength of Thornel-300 carbon single fibres tested using a gauge length of 60 mm. The solid line corresponds to the fitted beta distribution.

probability density function yields the beta distribution, which is also shown in Fig. 1.

In the beta distribution, the values of the variate are limited to a finite interval, which is more realistic for the strength of a brittle solid. It has also two shape parameters. On the other hand, the Weibull distribution has been criticised [9, 16] for its physically unsatisfactory boundary condition $\sigma = \infty$ for $F(\sigma) = 1$ (certainty of failure). To overcome this limitation Kies [16] proposed a modification of the flaw distribution function of the form

$$N(\sigma) = \left(\frac{\sigma - \sigma_L}{\sigma_U - \sigma}\right)^{m_0} \tag{3}$$

where $N(\sigma)$ is the number of flaws having strength σ or less, σ_L and σ_U are the lower and upper strength limit, respectively, and m_0 is defined as the damage coefficient. The functional form of Equation 3 is similar to one obtained from the beta distribution, except that it has only one shape parameter. Thus a further modification has been suggested [11, 12] in the form

$$N(\sigma) = \left(\frac{\sigma - \sigma_L}{\sigma_{01}}\right)^{m_1} \Bigg/ \left(\frac{\sigma_U - \sigma}{\sigma_{02}}\right)^{m_2} \tag{4}$$

where σ_{01}, σ_{02} and m_1, m_2 are the two scaling and shape parameters, respectively. For a brittle material like carbon fibre a lower strength limit $\sigma_L = 0$ and an upper strength limit equal to some realistic theoretical maximum value are reasonable [13]. Thus Equation 1 becomes

$$F(\sigma) = 1 - \exp\left[-L\left(\frac{\sigma}{\sigma_{01}}\right)^{m_1} \Bigg/ \left(\frac{\sigma_U - \sigma}{\sigma_{02}}\right)^{m_2}\right] \tag{5}$$

2.2. Fibre bundle tensile load–strain relationship

From Equations 5 and 2, we obtain

$$F(\varepsilon) = 1 - \exp\left[-L\left(\frac{\varepsilon}{\varepsilon_{01}}\right)^{m_1} \Bigg/ \left(\frac{\varepsilon_U - \varepsilon}{\varepsilon_{02}}\right)^{m_2}\right] \tag{6}$$

where $F(\varepsilon)$ is the failure probability for a strain ε and below; ε_U is the upper limit of failure strain and

$$\begin{aligned} \varepsilon_{01} &= \sigma_{01}/E_f \\ \varepsilon_{02} &= \sigma_{02}/E_f \end{aligned} \tag{7}$$

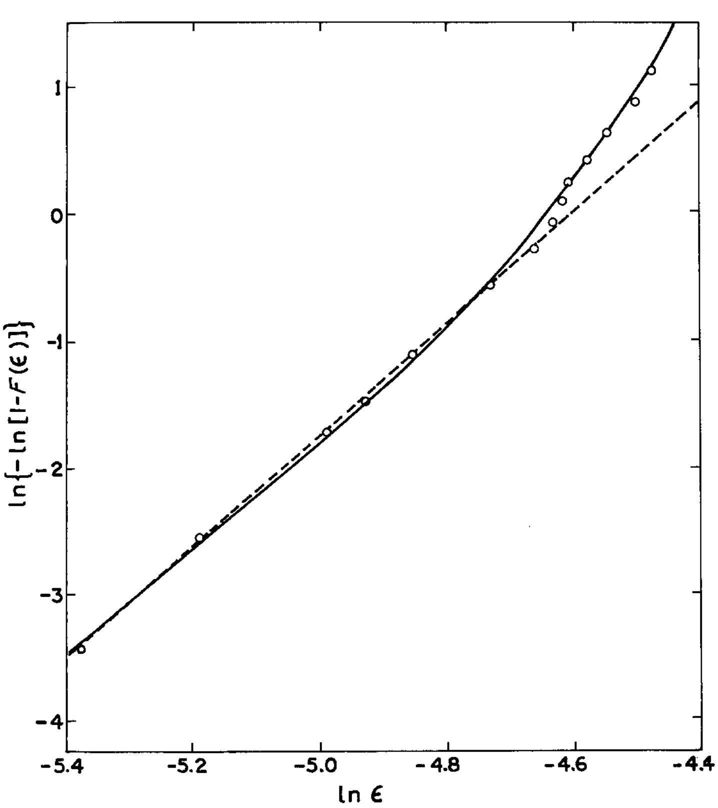

Figure 2 (○) Strength distribution of single fibres obtained from the tensile curve of a fibre bundle [10]. (– – –) Weibull, (——) proposed distribution.

For a bundle of fibres consisting of N_0 fibres, the number of surviving fibres at an applied strain ε is

$$N = N_0[1 - F(\varepsilon)] = N_0 \times \exp\left[-L\left(\frac{\varepsilon}{\varepsilon_{01}}\right)^{m_1} \bigg/ \left(\frac{\varepsilon_U - \varepsilon}{\varepsilon_{02}}\right)^{m_2}\right] \quad (8)$$

Thus, the applied load P on the bundle is given by

$$P = \sigma A N = A E_f \varepsilon N_0 \times \exp\left[-L\left(\frac{\varepsilon}{\varepsilon_{01}}\right)^{m_1} \bigg/ \left(\frac{\varepsilon_U - \varepsilon}{\varepsilon_{02}}\right)^{m_2}\right] \quad (9)$$

where A is the cross-sectional area of a single fibre in the bundle. Thus, if A, N_0, L, E_f, ε_{01}, ε_{02}, ε_U, m_1 and m_2 are known, the P–ε curve for a bundle of fibres could be drawn according to Equation 9.

Differentiating Equation 9 with respect to ε,

$$\frac{dP}{d\varepsilon} = A E_f N_0 \left[1 - \left(\frac{L m_1}{\varepsilon_{01}}\right)\left(\frac{\varepsilon}{\varepsilon_{01}}\right)^{m_1 - 1} \bigg/ \left(\frac{\varepsilon_U - \varepsilon}{\varepsilon_{02}}\right)^{m_2} - \left(\frac{L m_2}{\varepsilon_{02}}\right)\left(\frac{\varepsilon}{\varepsilon_{01}}\right)^{m_1} \bigg/ \left(\frac{\varepsilon_U - \varepsilon}{\varepsilon_{02}}\right)^{m_2 + 1}\right] \times \exp\left[-L\left(\frac{\varepsilon}{\varepsilon_{01}}\right)^{m_1} \bigg/ \left(\frac{\varepsilon_U - \varepsilon}{\varepsilon_{02}}\right)^{m_2}\right] \quad (10)$$

and the slope at $\varepsilon = 0$ is

$$S_0 = \left.\frac{dP}{d\varepsilon}\right|_{\varepsilon=0} = A E_f N_0 \quad (11)$$

which is identical to one obtained from the Weibull distribution [10]. Thus the equation for tangent line of the P–ε curve at $\varepsilon = 0$ is

$$P^* = A E_f N_0 \varepsilon \quad (12)$$

Combining Equations 9 and 12,

$$P/P^* = 1 - F(\varepsilon)$$

or

$$P/S_0\varepsilon = 1 - F(\varepsilon) \quad (13)$$

Thus the survival probability at any strain level ε on the P–ε curve can be evaluated from Equation 13 by calculating S_0 from Equation 11 with the data E_f, A and N_0 of the fibre bundle. Once the experimental values of ε and $[1 - F(\varepsilon)]$ are known, the parameters of the single-fibre strength distribution can be obtained by fitting Equation 6, which is written in the form

$$\ln\left[\left(\frac{1}{L}\right)\ln\left(\frac{1}{1 - F(\varepsilon)}\right)\right] = m_1 \ln\left(\frac{\varepsilon}{\varepsilon_{01}}\right) - m_2 \ln\left(\frac{\varepsilon_U - \varepsilon}{\varepsilon_{02}}\right) \quad (14)$$

3. Data analysis and discussion

The applicability of the method developed in this paper has been evaluated in terms of the data reported by Chi *et al.* [10] and Chi and Chou [14]. The data comprise single-fibre strengths measured using a gauge length of 60 mm and the load–strain (P–ε) curves for bundles of Thornel-300 carbon fibres ($N_0 = 1000$, fibre diameter $= 7\,\mu$m, $E_f = 225$ GPa) measured at the same gauge length. As reported by Chi *et al.* [10], Fig. 2 shows the experimental points for a fibre bundle plotted on Weibull axes.

For fitting Equation 14 to the data an initial estimate of ε_U was taken as 0.1 (assuming an approximate theoretical maximum strength equal to 1/10th of the elastic modulus). A set of values was assumed for ε_{01} and ε_{02} and the parameters m_1 and m_2 were evaluated from the experimental data by regression analysis. From the calculated and experimental values of $\ln\{(1/L)\ln[1/(1 - F(\varepsilon)]\}$, a least-squares sum was evaluated for the particular set of parameters ε_{01} and ε_{02}. The computation was then iterated with a new set of ε_{01} and ε_{02} until the minimum least-squares sum was found. The process was repeated by changing the value of ε_U until the minimum least-squares sum was obtained. The values of the parameters thus obtained are $\varepsilon_U = 0.016$, $\varepsilon_{01} = 0.08$, $\varepsilon_{02} = 0.02$, $m_1 = 2.995$ and $m_2 = 1.937$. Equation 14 is plotted in Fig. 2, showing excellent agreement with the data. Also plotted in Fig. 2 is the Weibull equation given by Chi *et al.* [10] with $m = 4.38$ and $\varepsilon_0 = 0.0257$.

In order to compare experimental data with the fitted distribution functions, the sum of squares is used as a measure of the goodness of fit between the function and data. The sum of squares is given by

$$Q = 1 - \frac{\sum_{i=1}^{n}(\varepsilon_i - \hat{\varepsilon}_i)^2}{\sum_{i=1}^{n}(\varepsilon_i - \bar{\varepsilon}_i)^2} \quad (15)$$

where $\hat{\varepsilon}_i$ is the strain value calculated for the appropriate F value and the calculated parameters of the distribution function; ε_i is the measured strain and $\bar{\varepsilon}_i$ is the mean of the distribution. For the Weibull distribution $\hat{\varepsilon}_i$ values are obtained from the expression

$$\hat{\varepsilon}_i^m = \varepsilon_0^m [1 - F(\varepsilon)]$$

For the proposed distribution $\hat{\varepsilon}_i$ is obtained by solving Equation 14 by the Newton–Raphson method. For a perfect fit between the data and the distribution function the value of Q will be equal to unity. In general $Q \gtrsim 0.95$ indicates a good fit. In the present case, the values of Q are obtained as 0.997 and 0.897 for the proposed (Equation 14) and Weibull distribution function, respectively, indicating that the Weibull distribution provides a poor fit to the data.

The theoretical P–ε curve calculated from Equation 5 with the parameters obtained before is shown in Fig. 3 by the solid line. The experimental data points are also indicated. The consistency between the theory and experiment is excellent in the entire range of strain. Also shown in Fig. 3 is the curve corresponding to the Weibull distribution.

In order to compare the values of parameters evaluated from bundle strength with single-fibre data, Equation 5 was fitted to the single-fibre data (60 mm gauge length) with $\sigma_U = \varepsilon_U E_f = 3600$ MPa, $\sigma_{01} = \varepsilon_{01} E_f = 18\,000$ MPa and $\sigma_{02} = \varepsilon_{02} E_f = 4500$ MPa. The regression analysis gives the values of $m_1 = 2.659$ and $m_2 = 0.423$. The value of Q is obtained as 0.969, indicating a good fit. The fitted equation along with the experimental data is shown in Fig. 4. It may be

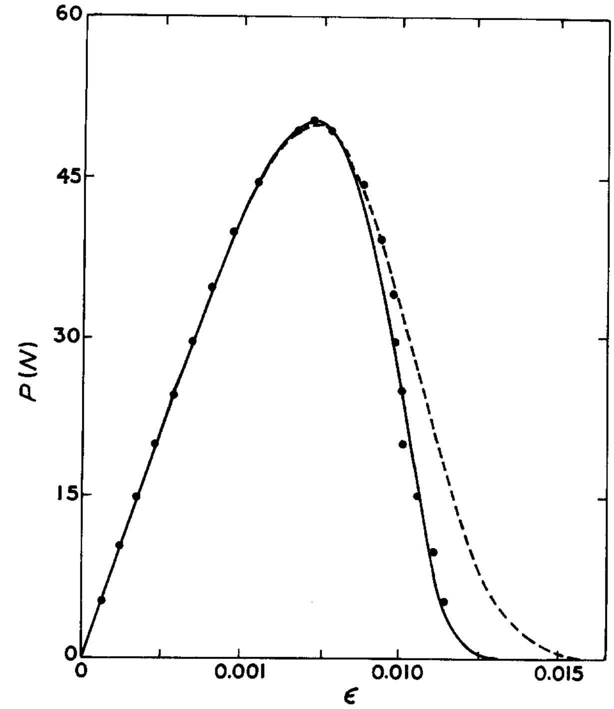

Figure 3 Comparison of theoretical P–ε curves with experimental data. (●) Experimental, (– – –) Weibull, (——) proposed.

noted that though the value of m_1 remains nearly identical for both the bundle and the single fibre, there is a marked change in the value of m_2 in the case of single fibres. Like the Weibull modulus, if we associate these parameters with scatter in the low- and high-strength groups of fibres, a low value of m_2 in the case of single fibres indicates a larger scatter in the high-strength group compared to that of the bundle. A possible explanation for this may be given as follows: the strength of a bundle will depend upon the proportion of high- and low-strength fibres present in it. For a bundle in which the majority of the fibres are of low strength the fibres will break at a comparatively low load, thereby increasing the load on a small number of high-strength fibres. Also, there will be a dynamic load on the surviving fibres due to sudden breakage of the low-strength group. Thus, the small number of high-strength fibres will contribute little to the strength and the bundle strength will be essentially controlled by the low-strength group, giving a similar distribution of strength to that of single fibres. On the other hand, for a bundle in which there is a small number of low-strength fibres, the breakage of these during loading will have comparatively less effect on the large number of surviving fibres, and the strength of the bundle will depend essentially on the high-strength group only. Thus the effect of comparatively low-strength fibres will be masked, showing a different distribution to that of single fibres.

It may be noted that the theoretical P–ε curve as shown in Fig. 3 is a continuous and smooth curve under both loading and unloading conditions. The experimental curves as given by Chi *et al.* [10] exhibit steps, i.e. decreases in tensile load at constant strain, which are attributed to the dynamic load effect.

4. Conclusions

1. The strength distribution of single fibres can be described by Equation 5. The function provides an upper and a lower strength limit, which is consistent with the boundary condition of the physical phenomena it represents.

2. The theoretical P–ε curve for a bundle of fibre (Equation 9) derived on the basis of above equation shows excellent agreement with the experimental data during both loading and unloading of the bundle.

3. Determination of the distribution parameters of single fibres from the bundle P–ε curve shows that reliable values of the parameters can be obtained except for the shape parameter m_2.

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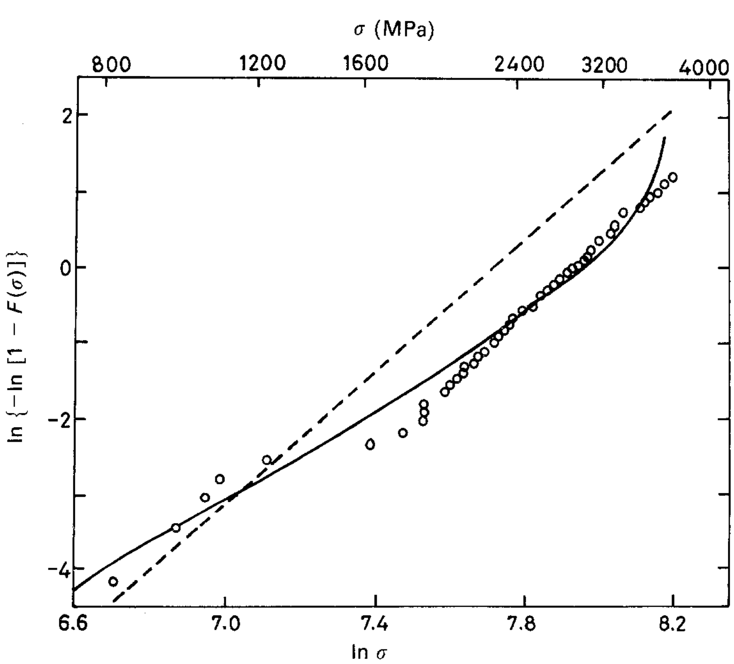

Figure 4 Strength distributions of single fibres (60 mm gauge length) plotted on Weibull axes. (○) Experimental, (– – –) Weibull, (——) proposed.

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